ON A CLASS OF MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED RUSCHEWEYH DERIVATIVES ASSOCIATED WITH FRACTIONAL DERIVATIVE OPERATOR

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ABSTRACT

By making use of generalized Ruscheweyh derivative operator we define a new class \( N_p(\lambda, \eta, \gamma) \) of multivalent functions associated with fractional derivative operator. We have derived some results on coefficient estimate, closure theorem, radius of starlikeness and convexity. Also we have applied general fractional derivative operator techniques to obtain some results for the functions belonging to this class.

Keywords  

1. INTRODUCTION

Let \( A(p) \) denote the class of analytic and p-valent functions of the form

\[
f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k
\]

in the open unit disc \( U = \{ z; z \in \mathbb{C}, |z| < 1 \} \). Where \( p \) is some positive integer.

Let \( f(z) \) is given by (1) and \( g(z) \) is given by

\[
g(z) = z^p - \sum_{k=n+p}^{\infty} \beta_k z^k
\]

We define convolution product (or hadamard product) of \( f \) and \( g \) by

\[
(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k \beta_k z^k
\]

Generalized fractional derivative operator of order \( \gamma \) is defined by Srivastava and Silverman [1,2]

\[
J_{0,t}^{\gamma,\mu,\theta} f(z) = \left[ \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \right] \int_0^t (z-t)^{-\gamma} f(z) \frac{d}{dz} \int_{0,t}^{\gamma-\mu,\theta} f(z),
\]

\[0 \leq \gamma < 1, \quad 0 \leq \gamma < n + 1, \quad n \in \mathbb{N}\]

Where \( f \) is an analytic function in a simply connected region of complex \( z \)-plane containing the origin, and the multiplicity of \( (z-t)^{-\gamma} \) is removed by requiring \( \log(z-t) \) to be real when \( z-t > 0 \). In term of gamma function, we have

\[
J_{0,t}^{\gamma,\mu,\theta} z^p = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \Gamma(\gamma, \theta+2) \Gamma(p-\mu, \gamma+2) z^{p-\mu}, \quad 0 \leq \gamma < 1, \quad p > \max\{0, \mu - \theta - 1\} - 1
\]

From the above definition we have \( D_z^\gamma f(z) \) fractional derivative operator of order \( \gamma \).

\[
J_{0,t}^{\gamma,\mu,\theta} f(z) = D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^t \frac{f(t)}{(z-t)^\gamma} dt, \quad 0 \leq \gamma < 1
\]

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In terms of gamma function we have

$$D_z^\gamma z^p = \frac{\Gamma(p+1)}{\Gamma(p+1)} z^{p-\gamma}, \quad (0 \leq \gamma)$$  \hspace{2cm} (7)

Fractional integral operator of order $\gamma$ is also given by

$$D_z^{-\gamma} f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{f(t)}{(z-t)^{1-\gamma}} \, dt,$$  \hspace{2cm} (8)

Where $f(z)$ is an analytic function in a simply-connected region of the complex $z$-plane containing the origin and the multiplicity of $(z-t)^{\gamma-1}$ is removed by requiring $\log(z-t)$ to the real when $z-t > 0$. In terms of gamma function, we have

$$D_z^{-\gamma} z^p = \frac{\Gamma(p+1)}{\Gamma(p+\gamma+1)} z^{p+\gamma}$$  \hspace{2cm} (9)

Goyal [2] defined generalized Ruscheweyh derivative as follows:

$$j_p^\gamma f(z) = \frac{\Gamma(p+\gamma+2)}{\Gamma(\gamma+2)!} (z^p f(z)) = z^p - \sum_{k=n+p}^{\infty} a_k A_p^\mu(k) z^k$$  \hspace{2cm} (10)

where

$$A_p^\mu(k) = \frac{\Gamma(k-p+1) \Gamma(k+\delta-p+2)}{\Gamma(k-p+1) \Gamma(k+\delta+2+p)}$$  \hspace{2cm} (11)

For $\gamma = \mu$ generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of $f(z)$ of order $\gamma$ [4]

$$D^\gamma f(z) = \frac{z^p}{\Gamma(\gamma+1)} \frac{d}{dz} \left( z^{\gamma-p} f(z) \right) = z^p + \sum_{k=n+p}^{\infty} \alpha_k A_k(\gamma) z^k$$  \hspace{2cm} (12)

where

$$A_k(\gamma) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma+p) \Gamma(k-p+1)}$$  \hspace{2cm} (13)

**Definition:** A function is said to be in the class $N_p(\lambda, \eta, \gamma)$, if it satisfies the condition

$$\text{Re} \left\{ \frac{z \left( j_p^\mu f(z) \right)}{(1-\lambda) \left( j_p^\mu f(z) \right)} \right\} > \eta$$  \hspace{2cm} (14)

for $z \in U$ and $0 \leq \lambda < 1, 0 \leq \eta < p$ and $\gamma > -1$.

2. **COEFFICIENT ESTIMATE**

**Theorem 1:** Let $f(z) \in A(p)$, $z \in U$ be of the form (1). Then $f(z) \in N_p(\lambda, \eta, \gamma)$ if

$$\sum_{k=n+p}^{\infty} \frac{(k-\eta)(1-\lambda)(k-1)}{(p-\eta)(1-\lambda)(p-1)} \alpha_k A_p^\mu(k) < 1$$  \hspace{2cm} (15)

where $0 \leq \lambda < 1, 0 \leq \eta < p$ and $\gamma > -1$.

Proof: Suppose that $f(z) \in N_p(\lambda, \eta, \gamma)$ then

$$\text{Re} \left\{ \frac{z \left( j_p^\mu f(z) \right)}{(1-\lambda) \left( j_p^\mu f(z) \right)} \right\} > \eta$$

by using (10) in the above equation, we get

$$\text{Re} \left\{ \frac{z \left( j_p^\mu f(z) \right)}{(1-\lambda) \left( j_p^\mu f(z) \right)} \right\} > \eta$$

on letting $|z| < r \to 1^-$ we obtain

$$\sum_{k=n+p}^{\infty} \frac{(k-\eta)(1-\lambda)(k-1)}{(p-\eta)(1-\lambda)(p-1)} \alpha_k A_p^\mu(k) < 1$$

This proves the theorem.
3. CLOSURE THEOREMS

**Theorem 2:** Let \( b_j \geq 0 \) for \( j \in \{1,2,3,\ldots,m\} \) and \( \sum_{j=1}^{m} b_j = 1 \) the function \( f_{j}(z) \in N_p(\lambda, \eta, \gamma) \) for \( j=1,2,3,\ldots,m \) where \( f_j(z) \) is defined as

\[
f_{j}(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_{k,j} z^k \quad (\alpha_{k,j} \geq 0)
\]

then the function \( H(z) \) is also in the class \( N_p(\lambda, \eta, \gamma) \) where

\[
H(z) = \sum_{j=1}^{m} b_j f_j(z) = z^p - \sum_{k=n+p}^{\infty} \left[ \sum_{j=1}^{m} b_j \alpha_{k,j} \right] z^k
\]

**Proof:** For \( j \in \{1,2,3,\ldots,m\} \), we obtain

\[
\sum_{j=1}^{m} b_j f_j(z) = z^p - \sum_{k=n+p}^{\infty} \sum_{j=1}^{m} b_j \alpha_{k,j} z^k
\]

hence

\[
\sum_{k=n+p}^{\infty} \left[ \sum_{j=1}^{m} b_j \alpha_{k,j} \right] z^k < 1
\]

this proves the theorem.

**Theorem 3:** Let \( f(z) \) defined by (1) and \( g(z) \) defined by (2) are in the class \( N_p(\lambda, \eta, \gamma) \). Then the function

\[
h(z) = z^p - \sum_{k=n+p}^{\infty} (\alpha_k^2 + \beta_k^2) z^k
\]

is also in the class \( N_p(\lambda, \eta, \gamma_1) \) where

\[
\gamma_1 \leq \inf_k \left[ \frac{(k-p)(k-n)(1-\lambda(k-1))}{2(p-n)(1-\lambda(p-1))} \right] \left( A_p^{\gamma_1}(k) \right)^2 - 1
\]

**Proof:** Since \( f, g \in N_p(\lambda, \eta, \gamma) \),

\[
\sum_{k=n+p}^{\infty} \left[ \frac{(k-n)(1-\lambda(k-1))}{(p-n)(1-\lambda(p-1))} A_p^{\gamma_1}(k) \right] \alpha_k^2 \\
\leq \sum_{k=n+p}^{\infty} \left[ \frac{(k-n)(1-\lambda(k-1))}{(p-n)(1-\lambda(p-1))} A_p^{\gamma_1}(k) \right] \alpha_k^2 < 1
\]

similarly

\[
\sum_{k=n+p}^{\infty} \left[ \frac{(k-n)(1-\lambda(k-1))}{(p-n)(1-\lambda(p-1))} A_p^{\gamma_1}(k) \right] \beta_k^2 < 1
\]

therefore

\[
\sum_{k=n+p}^{\infty} \frac{1}{2} \left[ \frac{(k-n)(1-\lambda(k-1))}{(p-n)(1-\lambda(p-1))} A_p^{\gamma_1}(k) \right] \left( \alpha_k^2 + \beta_k^2 \right) < 1
\]

now we must show that

\[
\sum_{k=n+p}^{\infty} \left[ \frac{(k-n)(1-\lambda(k-1))}{(p-n)(1-\lambda(p-1))} A_p^{\gamma_1}(k) \right] \left( \alpha_k^2 + \beta_k^2 \right) < 1
\]

this inequality holds if
\[
\frac{(k - \eta)(1 - \lambda(k - 1))}{(p - \eta)(1 - \lambda(p - 1))} A_p^{Y_{1,\mu}}(k) \leq \frac{1}{2} \left( \frac{(k - \eta)(1 - \lambda(k - 1))}{(p - \eta)(1 - \lambda(p - 1))} A_p^{Y_{\mu}}(k) \right)^2
\]

which is equivalent to

\[
A_p^{Y_{1,\mu}}(k) = \frac{(k - \eta)(1 - \lambda(k - 1))}{2(p - \eta)(1 - \lambda(p - 1))} [A_p^{Y_{\mu}}(k)]^2
\]

since \( A_p^{Y_{1,\mu}}(k) \geq \frac{y_1 + 1}{k - p} \), we obtain

\[
y_1 + 1 \leq \frac{1}{2} \left( \frac{(k - \eta)(1 - \lambda(k - 1))}{(p - \eta)(1 - \lambda(p - 1))} A_p^{Y_{\mu}}(k) \right)^2
\]

this gives required result.

4. RADIUS OF STARLIKENESS AND CONVEXITY

**Theorem 3:** Let the function \( f_p^{Y_{\mu}}(z) \) defined by (10) then \( f_p^{Y_{\mu}}(z) \) is starlike of order \( \delta, \leq \delta < 1 \) in disc \(|z| < R_1\) where

\[
R_1 = \inf_k \left\{ \frac{(1-\delta)(k-\eta)(1-\lambda(k-1))}{(k-p+\delta)(p-\eta)(1-\lambda(p-1))} A_p^{Y_{\mu}}(k) \right\}^{\frac{1}{k-p}} \tag{21}
\]

**Proof:** It is enough to show that

\[
\left| \frac{z(f_p^{Y_{\mu}}(z))'}{f_p^{Y_{\mu}}(z)} - p \right| < 1 - \delta \text{ for } |z| < R_1
\]

where \( R_1 \) is given by (21). By using (10) we have

\[
\left| \frac{z(f_p^{Y_{\mu}}(z))'}{f_p^{Y_{\mu}}(z)} - p \right| \leq \frac{\sum_{k=n+p}^{\infty} (k-p) \alpha_k A_p^{Y_{\mu}}(k) |z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} \alpha_k A_p^{Y_{\mu}}(k) |z|^{k-p}}
\]

thus we have

\[
\left| \frac{z(f_p^{Y_{\mu}}(z))'}{f_p^{Y_{\mu}}(z)} - p \right| < 1 - \delta
\]

if

\[
\sum_{k=n+p}^{\infty} \frac{(k-p+1-\delta) \alpha_k A_p^{Y_{\mu}}(k) |z|^{k-p}}{1-\delta} < 1
\]

but by theorem 1 above equation will be true if

\[
\frac{(k - p + 1 - \delta)}{1 - \delta} |z|^{k-p} \leq \frac{(k - \eta)(1 - \lambda(k - 1))}{(p - \eta)(1 - \lambda(p - 1))} A_p^{Y_{\mu}}(k)
\]

that is if

\[
|z| \leq \left( \frac{(1 - \delta)(k - \eta)(1 - \lambda(k - 1))}{(k - p + 1 - \delta)(p - \eta)(1 - \lambda(p - 1))} A_p^{Y_{\mu}}(k) \right)^{\frac{1}{k-p}}
\]

This proves theorem.

**Theorem 4:** Let the function \( f_p^{Y_{\mu}}(z) \) defined by (10) then \( f_p^{Y_{\mu}}(z) \) is convex of order \( \delta, 0 \leq \delta < 1 \) in disc \(|z| < R_2\) where

\[
R_2 = \inf_k \left\{ \frac{(p-\delta)(k-\eta)(1-\lambda(k-1))}{k(k-\delta)(p-\eta)(1-\lambda(p-1))} A_p^{Y_{\mu}}(k) \right\}^{\frac{1}{k-p}} \tag{23}
\]

**Proof:** It is enough to show that

\[
\left| \frac{z(f_p^{Y_{\mu}}(z))''}{(f_p^{Y_{\mu}}(z))'} - 1 \right| \leq 1 - \delta \text{ for } |z| < R_2
\]

where \( R_2 \) is given by (23). By using (10) we have
thus
\[
\left| \frac{z(\gamma_p^\mu f(z)))''}{(\gamma_p^\mu f(z))'} \right| \leq 1 - \delta
\]

if
\[
\sum_{k=n+p}^\infty k(k - \delta) \alpha_k A_p^\mu(k) |z|^{k-p} \leq 1
\]

by theorem 1 above equation will be true if
\[
\frac{k(k - \delta)}{p(p - \delta)} |z|^{k-p} \leq \frac{(k - \eta)[1 - \lambda(k - 1)]}{(p - \eta)[1 - \lambda(p - 1)]} A_p^\mu(k)
\]

that is if
\[
|z| \leq \frac{\left\{ p(p - \delta)(k - \eta)[1 - \lambda(k - 1)] \right\}^{\frac{1}{k - \delta} A_p^\mu(k)}}{k(k - \delta)(p - \eta)[1 - \lambda(p - 1)]^{\frac{1}{k - \delta} A_p^\mu(k)}}
\]

This proves theorem.

5. APPLICATION TO FRACTIONAL CALCULUS

Theorem 5:- Let \( f(z) \in N_p(\lambda, \eta, \gamma), \gamma \geq 0 \) then
\[
|z|^{p+\beta} \left\{ \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)} \right\} [1 - A|z|^n] \leq |D_z^{-\beta} f(z)| \leq |z|^{p+\beta} \left\{ \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)} \right\} [1 + A|z|^n]
\]

where
\[
A = \frac{(p + 1)[(p - \eta)[1 - \lambda(p - 1)](\mu - \gamma + \theta + 2,)(n + 1)\Gamma(n + 1)}}{(\beta + p + 1)[(n + p - \eta)[1 - \lambda(n + p - 1)](\mu + 1)(\theta + 2,)(n + 1)\Gamma(n + 1)}}
\]

and \( f(z) \) is analytic function.

Proof:- By equation (9) we have
\[
\frac{\Gamma(p+\beta+1)}{\Gamma(p+1)} z^{-\beta} D_z^{-\beta} f(z) = z^p - \sum_{k=n+p}^\infty \alpha_k M_p(k, \beta) z^k
\]

where
\[
M_p(k, \beta) = \frac{\Gamma(\beta+p+1)\Gamma(\gamma+1)}{\Gamma(\beta+p+1)\Gamma(1+1)}
\]

but \( A_p^\mu(k) \) is increasing function of \( k \), and also \( M_p(k, \beta) \) is a decreasing function for \( k \geq n+p \), thus, we have
\[
A_p^\mu(k) \geq \frac{(\mu+1)n(\theta+2,)(n+1)\Gamma(n+1)}}{\mu-\gamma+\theta+2,)(n+1)\Gamma(n+1)}
\]

and
\[
M_p(k, \beta) \leq \frac{\Gamma(\beta+p+1)\Gamma(n+1)\Gamma\left(n+\beta+1,\beta+1\right)}{\Gamma\left(n\beta+\beta+1,\beta+1\right)\Gamma\left(n+1\right)} = \frac{(p+1)n}{(\beta+p+1)n}
\]

from (25) we conclude that
\[
\left| \frac{\Gamma(p+\beta+1)}{\Gamma(p+1)} z^{-\beta} D_z^{-\beta} f(z) \right| \leq |z|^p + \frac{(p+1)n}{(\beta+p+1)n} |z|^{n+p} \sum_{k=n+p}^\infty \alpha_k
\]

\[
|z|^{-\beta} D_z^{-\beta} f(z) \leq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)} \left\{ 1 + A|z|^n \right\}
\]

where
\[ A = \frac{(p+1)n(p-\eta)(1-\lambda(p-1))(\mu-\gamma+\theta+2)a_n\Gamma(n+1)}{(\beta+p+1)n(n+p-\eta)(1-\lambda(n+p-1))(\mu+1)a_n(\theta+2)n} \]

thus we get
\[ |D_z^{-\beta}f(z)| \leq |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)} [1 + A|z|^p] \tag{30} \]

also
\[ |D_z^{-\beta}f(z)| \geq |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)} [1 - A|z|^p] \tag{31} \]

**Theorem 6:** Let \( f(z) \in N_p(\lambda, \eta, \gamma), \gamma \geq 0 \) then
\[ |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} [1 - B|z|^n] \leq |D_z^{-\beta}f(z)| \leq |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} [1 + B|z|^n] \tag{32} \]

where
\[ B = \frac{(p+1)n(p-\eta)(1-\lambda(p-1))(\mu-\gamma+\theta+2)a_n\Gamma(n+1)}{(p-\beta+1)n(n+p-\eta)(1-\lambda(n+p-1))(\mu+1)a_n(\theta+2)n} \]

and \( f(z) \) is an analytic function

**Proof:** By using equation (7), we have
\[ \frac{\Gamma(n-\beta+1)}{\Gamma(p+1)} z^\beta D_z^\beta f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k B_p(k, \beta) z^k \tag{33} \]

where
\[ B_p(k, \beta) = \frac{\Gamma(p-\beta+1)\Gamma(k+1)}{\Gamma(k-\beta+1)\Gamma(p+1)} \tag{34} \]

but \( A_p^\mu(k) \) is increasing function of \( k \), and also \( B_p(k, \beta) \) is a decreasing function for \( k \geq n+p \), thus, we have
\[ A_p^\mu(k) \geq \frac{(\mu+1)n(\theta+2)n}{(\mu-\gamma+\theta+2)n}\Gamma(n+1) \tag{35} \]

and
\[ B_p(k, \beta) \leq \frac{\Gamma(p-\beta+1)\Gamma(n+p+1)}{\Gamma(n+p-\beta+1)\Gamma(p+1)} = \frac{(p+1)a_n}{(p-\beta+1)n} \tag{36} \]

from (33) we conclude that
\[ \left| \frac{\Gamma(p-\beta+1)}{\Gamma(p+1)} z^\beta D_z^\beta f(z) \right| \leq |z|^p + \frac{(p+1)\Gamma(n+1)}{(p-\beta+1)n} \sum_{k=n+p}^{\infty} \alpha_k \left| z \right|^n \tag{37} \]

where
\[ B = \frac{(p+1)n(p-\eta)(1-\lambda(p-1))(\mu-\gamma+\theta+2)a_n\Gamma(n+1)}{(p-\beta+1)n(n+p-\eta)(1-\lambda(n+p-1))(\mu+1)a_n(\theta+2)n} \]

thus we get
\[ |D_z^\beta f(z)| \leq |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} [1 + B|z|^n] \tag{38} \]

also
\[ |D_z^\beta f(z)| \geq |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} [1 - B|z|^n] \tag{39} \]
Corollary 1:- Letting $\beta = 1$ in theorem 5 we obtain

$$\frac{|z|^{p+1}}{(p+1)}[1-A|z|^n] \leq \left| \int_0^z f(t)dt \right| \leq \frac{|z|^{p+1}}{(p+1)}[1+A|z|^n]$$

where

$$A = \frac{(p-n+2)(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\theta+2)_n\Gamma(n+1)}{(p+2)(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\theta+2)_n}$$

and $f(z)$ is analytic function

Corollary 2:- Letting $\beta = 0$ in theorem 6 we obtain

$$|z|^p[1-B|z|^n] \leq |f(z)| \leq |z|^p[1+B|z|^n]$$

where

$$B = \frac{(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\theta+2)_n\Gamma(n+1)}{(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\theta+2)_n}$$

and $f(z)$ is analytic function

6. REFERENCES


